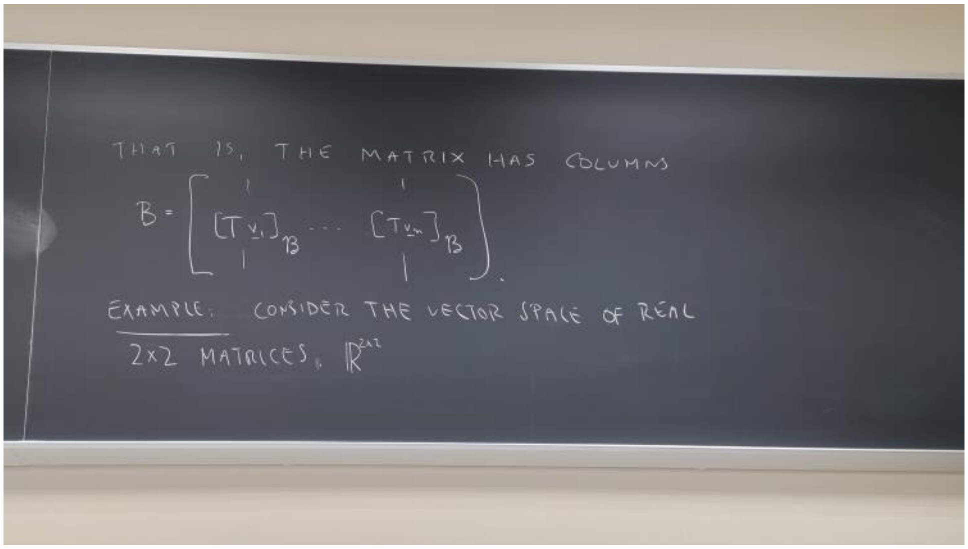
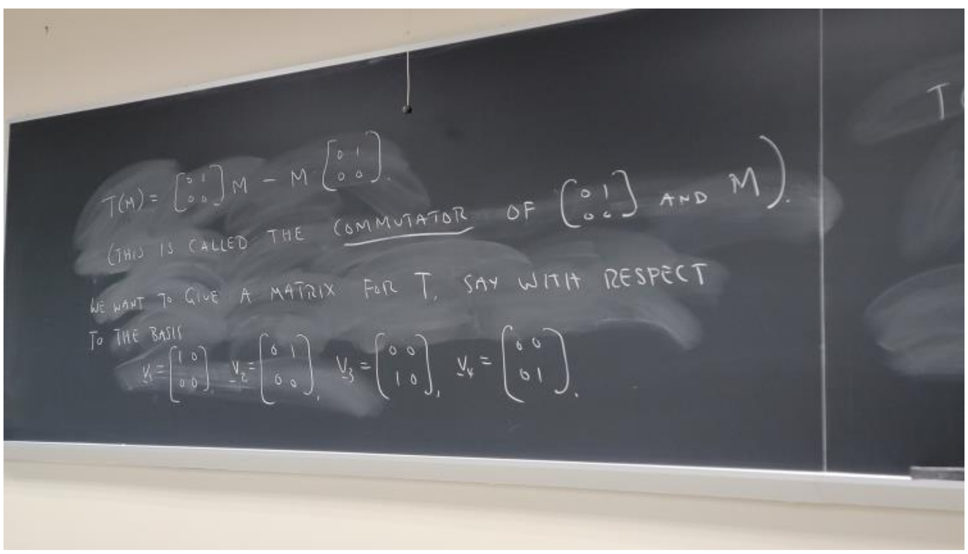
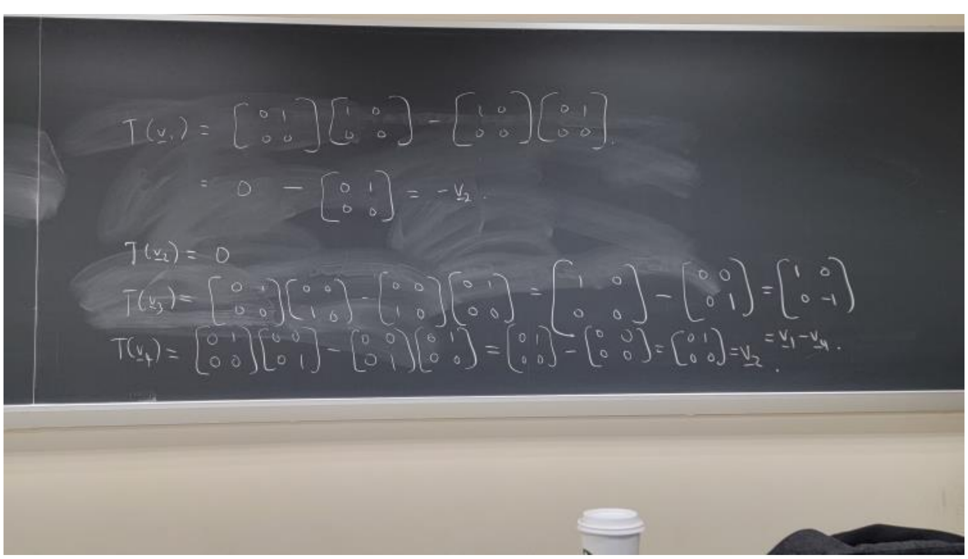
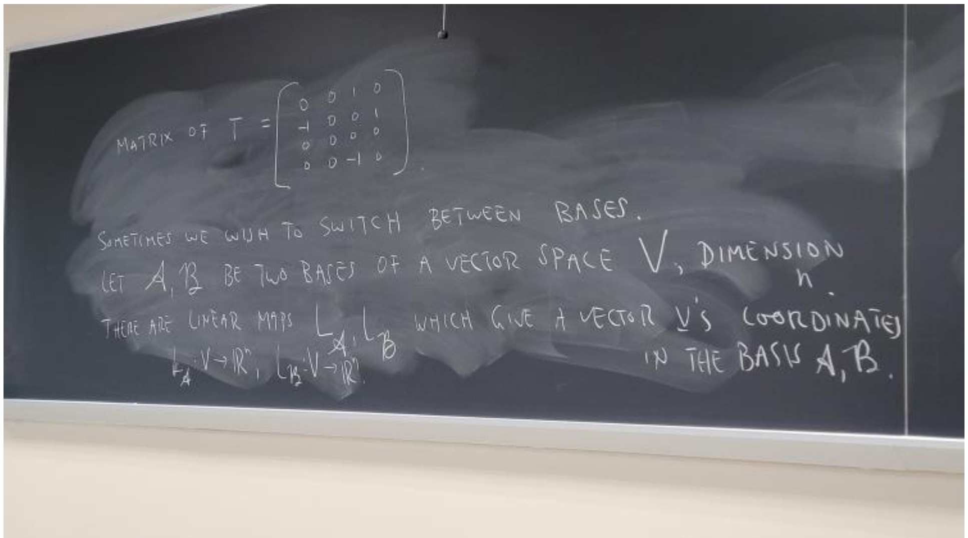
(2/28/23)

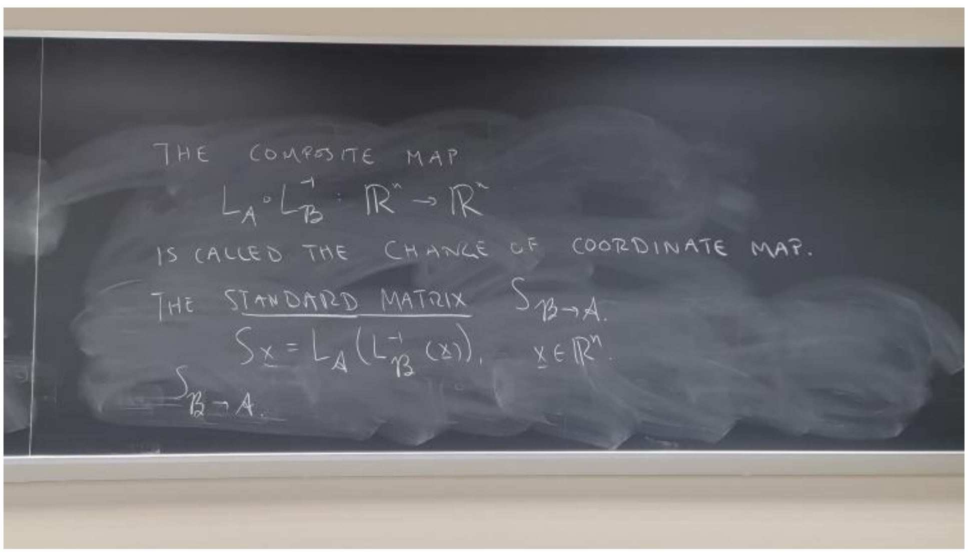
Thursday, March 2, 2023 10:30 AM

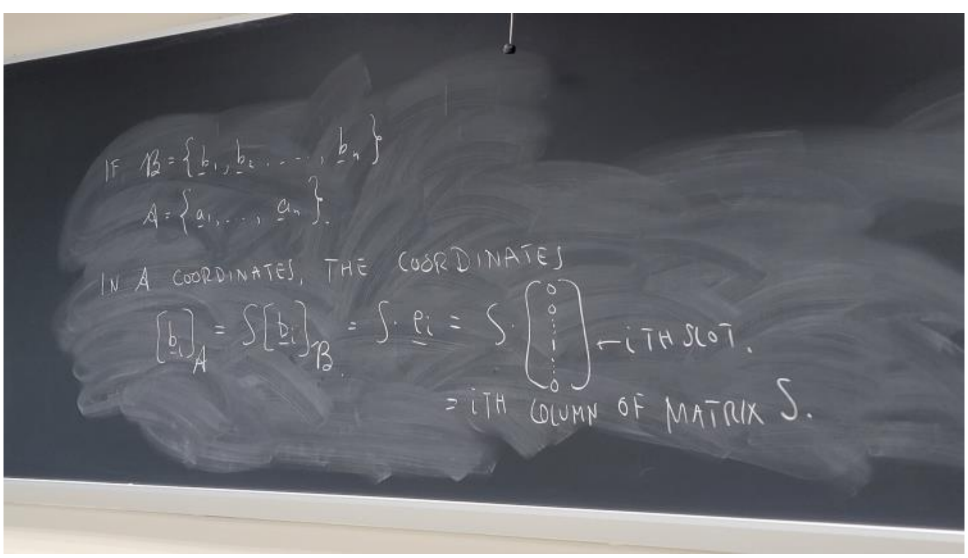







MATRIX OF $T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.
SOMETIMES WE WISH TO SWITCH BETWEEN BASES.
LET A, B BE TWO BASES OF A VECTOR SPACE V, DIMENSION n.
THERE ARE LINEAR MAPS L_A, L_B WHICH GIVE A VECTOR $\underline{v}$'s COORDINATES IN THE BASIS A, B.
$L_A : V \to \mathbb{R}^n$, $L_B : V \to \mathbb{R}^n$.


THE COMPOSITE MAP
$L_A \circ L_B^{-1} : \mathbb{R}^n \to \mathbb{R}^n$
IS CALLED THE CHANGE OF COORDINATE MAP.
THE STANDARD MATRIX $S_{B \to A}$.
$S\underline{x} = L_A(L_B^{-1}(\underline{x}))$, $\underline{x} \in \mathbb{R}^n$.
$S_{B \to A}$.


IF $B = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$
$A = \{\underline{a}_1, \dots, \underline{a}_n\}$.
IN A COORDINATES, THE COORDINATES
$[\underline{b}_i]_A = S[\underline{b}_i]_B = S \cdot \underline{e}_i = S \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ iTH SLOT.
= iTH COLUMN OF MATRIX S.

THIS MEANS THE MATRIX

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} | & | & & | \\ [\underline{b}_1]_{\mathcal{A}} & [\underline{b}_2]_{\mathcal{A}} & \cdots & [\underline{b}_n]_{\mathcal{A}} \\ | & | & & | \end{bmatrix}.$$

<u>EXAMPLE:</u> $V = \{x_1 + x_2 + x_3 = 0 : (x_1, x_2, x_3) \in \mathbb{R}^3\}$.

FIRST BASIS $\mathcal{A} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \right\}$.

$$S_{\mathcal{B}\to\mathcal{A}} = \left\{ [\underline{b}_1]_{\mathcal{A}}, [\underline{b}_2]_{\mathcal{A}} \right\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \qquad \square$$

<u>THEOREM.</u> $V < \mathbb{R}^n$ BASES. $\mathcal{A} = \{\underline{a}_1, \ldots, \underline{a}_m\}$, $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_m\}$.

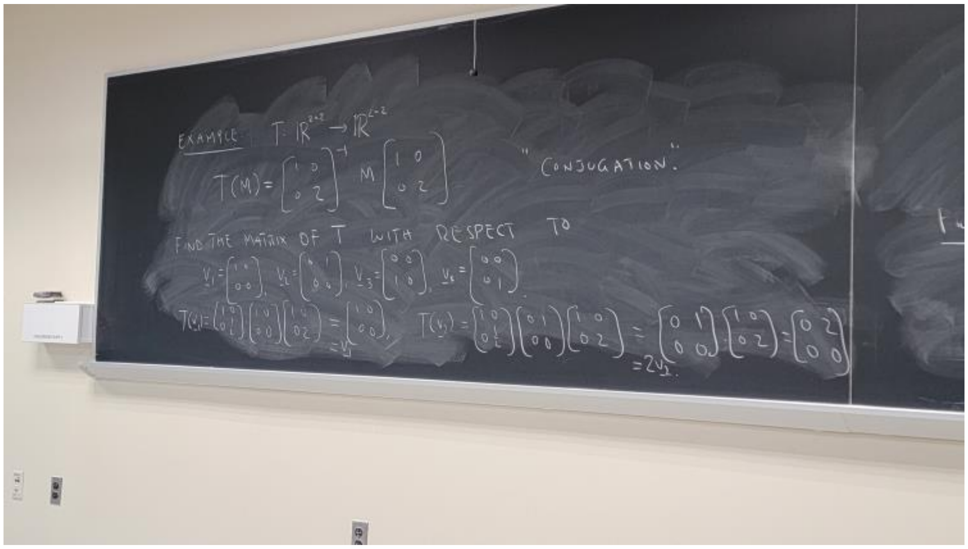
S CHANGE OF BASIS $\mathcal{B}$ TO $\mathcal{A}$

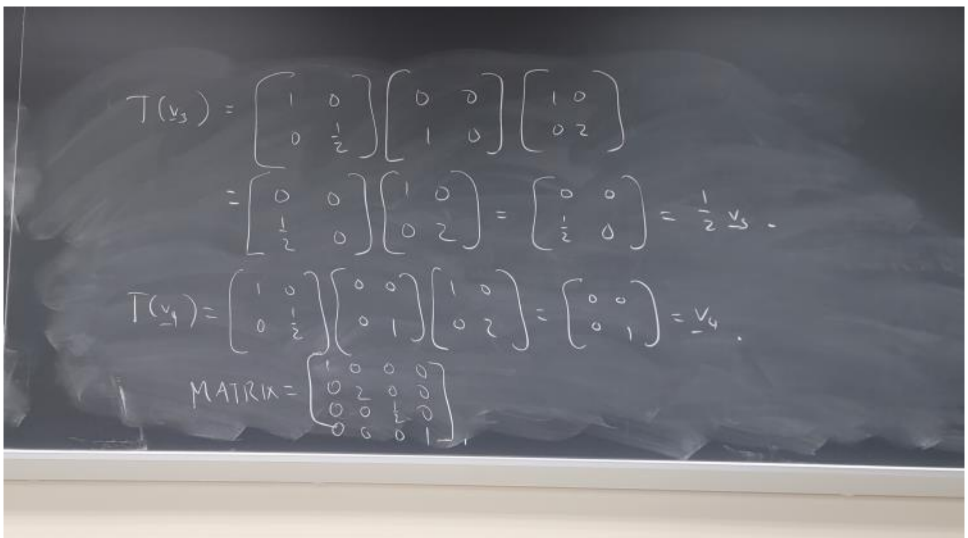
$$\begin{bmatrix} | & | & & | \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_m \\ | & | & & | \end{bmatrix} S_{\mathcal{B}\to\mathcal{A}}.$$

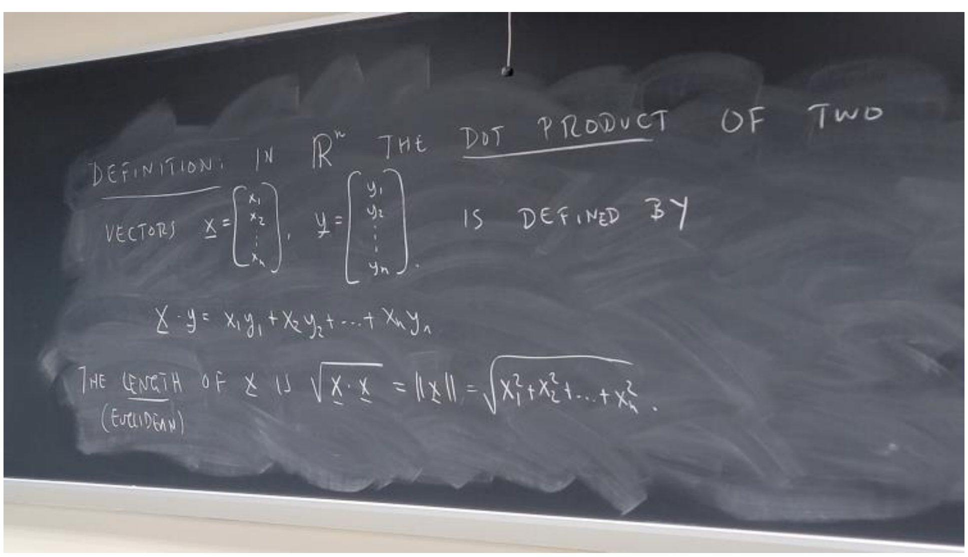
<u>PROOF:</u> THE iTH COLUMN $S_{\mathcal{B}\to\mathcal{A}}$ HAS COEFFICIENTS

$\begin{pmatrix} S_{1i} \\ S_{2i} \\ \vdots \\ S_{mi} \end{pmatrix}$ SO THAT $\underline{b}_i = S_{1i}\underline{a}_1 + S_{2i}\underline{a}_2 + \cdots + S_{mi}\underline{a}_m$

THE THEOREM FOLLOWS BECAUSE $A \cdot \begin{bmatrix} S_{1i} \\ \vdots \\ S_{mi} \end{bmatrix} = S_{1i}\underline{a}_1 + \cdots + S_{mi}\underline{a}_m = \underline{b}_i$. $\square$


EXAMPLE $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$
$T(M) = \begin{bmatrix}1&0\\0&2\end{bmatrix}^{-1} M \begin{bmatrix}1&0\\0&2\end{bmatrix}$
"CONJUGATION".
FIND THE MATRIX OF T WITH RESPECT TO
$\underline{v}_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}, \underline{v}_2 = \begin{bmatrix}0&1\\0&0\end{bmatrix}, \underline{v}_3 = \begin{bmatrix}0&0\\1&0\end{bmatrix}, \underline{v}_4 = \begin{bmatrix}0&0\\0&1\end{bmatrix}.$
$T(\underline{v}_1) = \begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{=\underline{v}_1},$
$T(\underline{v}_2) = \begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \underbrace{\begin{bmatrix}0&2\\0&0\end{bmatrix}}_{=2\underline{v}_2}.$


$T(\underline{v}_3) = \begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}$
$= \begin{bmatrix}0&0\\\frac{1}{2}&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \begin{bmatrix}0&0\\\frac{1}{2}&0\end{bmatrix} = \frac{1}{2}\underline{v}_3.$
$T(\underline{v}_4) = \begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix} = \underline{v}_4.$
MATRIX $= \begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{bmatrix}.$


DEFINITION: IN $\mathbb{R}^n$ THE DOT PRODUCT OF TWO
VECTORS $\underline{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}, \underline{y} = \begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}$. IS DEFINED BY
$\underline{x}\cdot\underline{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$
THE LENGTH OF $\underline{x}$ IS $\sqrt{\underline{x}\cdot\underline{x}} = \|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$
(EUCLIDEAN)

WE SAY TWO VECTORS $\underline{x}, \underline{y}$ IN $\mathbb{R}^n$ ARE <u>ORTHOGONAL</u> OR <u>PERPENDICULAR</u> IF $\underline{x} \cdot \underline{y} = 0$

$\underline{x}$ IS A <u>UNIT VECTOR</u> IF $\|\underline{x}\| = 1$.

THE DOT PRODUCT GIVES A NOTION OF <u>LENGTHS</u> AND <u>ANGLES</u> IN VECTOR SPACES.

<u>DEFINITION</u>. A SEQUENCE OF VECTORS ARE <u>ORTHONORMAL</u>, $\underline{v}_1, \dots, \underline{v}_n$ IF

$\|\underline{v}_1\| = \|\underline{v}_2\| = \dots = \|\underline{v}_n\| = 1$.

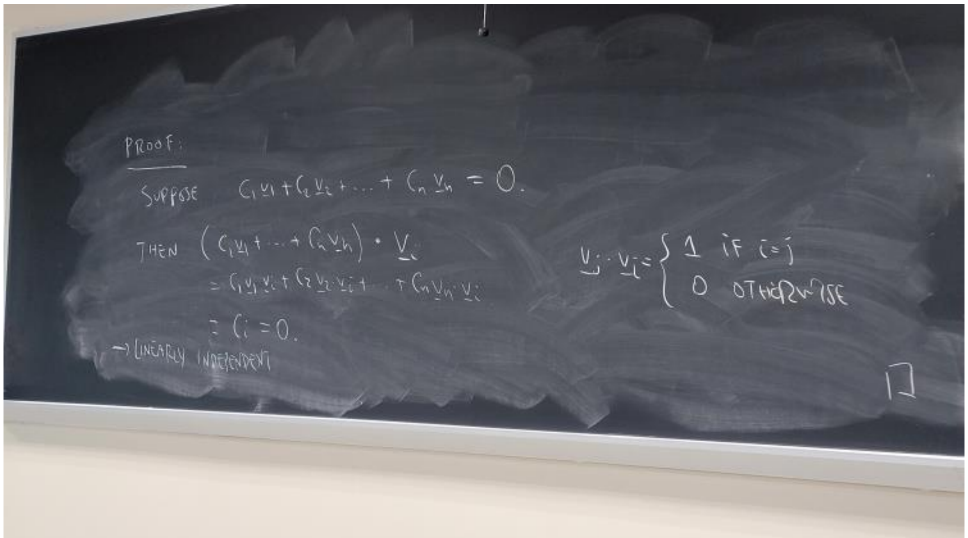
AND $\underline{v}_i \cdot \underline{v}_j = 0$ IF $i \neq j$.

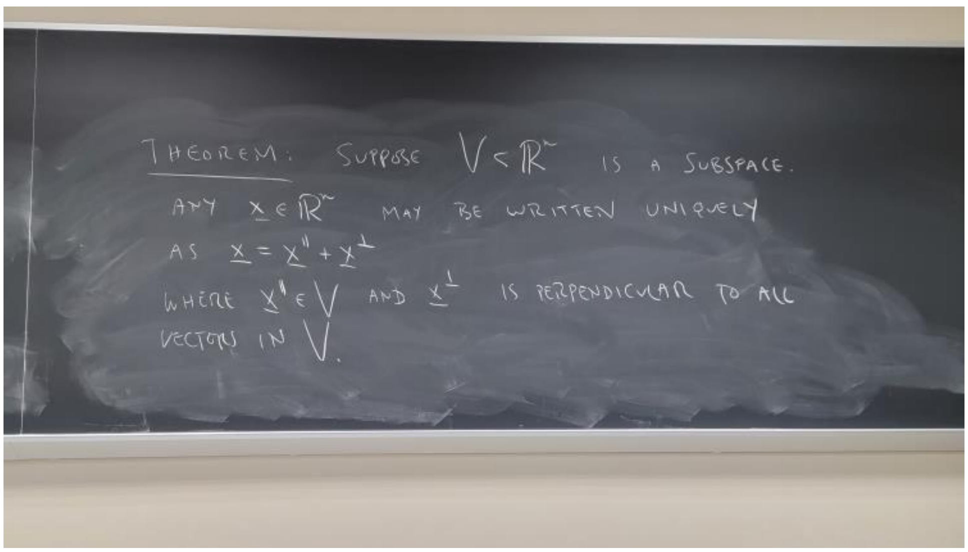
<u>EXAMPLE</u>: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

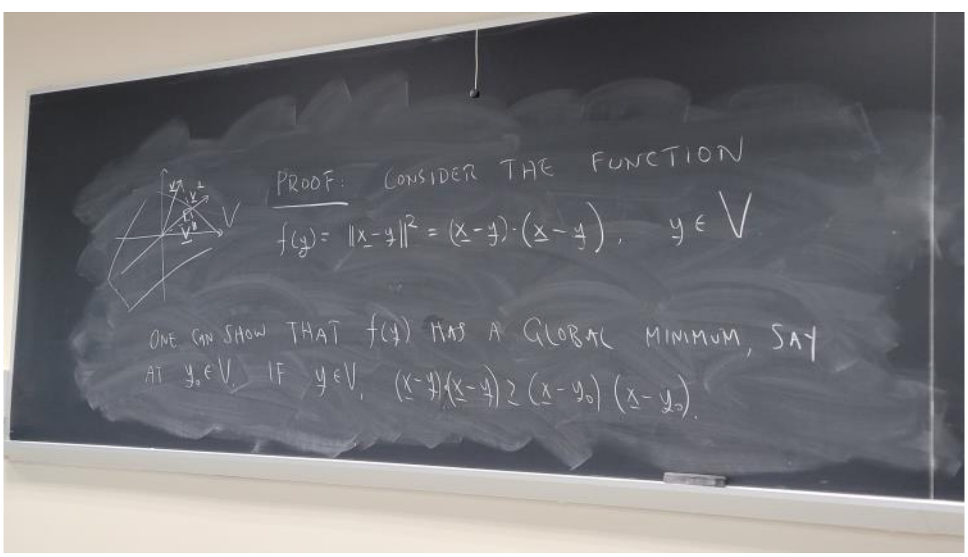
<u>EX</u>: $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

TWO ORTHONORMAL BASES FOR $\mathbb{R}^4$.

<u>THEOREM</u>: IF $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ ARE AN ORTHONORMAL LIST, THEY ARE LINEARLY INDEPENDENT. IF n VECTORS IN $\mathbb{R}^n$ ARE ORTHONORMAL, THEY ARE A BASIS.


PROOF:
SUPPOSE $c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n = 0$.
THEN $(c_1\underline{v}_1 + \dots + c_n\underline{v}_n) \cdot \underline{v}_i$
$= c_1\underline{v}_1\cdot\underline{v}_i + c_2\underline{v}_2\cdot\underline{v}_i + \dots + c_n\underline{v}_n\cdot\underline{v}_i$
$= c_i = 0$.
→ LINEARLY INDEPENDENT
$\underline{v}_j \cdot \underline{v}_i = \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{OTHERWISE} \end{cases}$
$\square$


THEOREM: SUPPOSE $V < \mathbb{R}^n$ IS A SUBSPACE.
ANY $\underline{x} \in \mathbb{R}^n$ MAY BE WRITTEN UNIQUELY
AS $\underline{x} = \underline{x}^{\parallel} + \underline{x}^{\perp}$
WHERE $\underline{x}^{\parallel} \in V$ AND $\underline{x}^{\perp}$ IS PERPENDICULAR TO ALL
VECTORS IN V.


V
PROOF: CONSIDER THE FUNCTION
$f(\underline{y}) = \|\underline{x} - \underline{y}\|^2 = (\underline{x} - \underline{y}) \cdot (\underline{x} - \underline{y})$, $\underline{y} \in V$
ONE CAN SHOW THAT $f(\underline{y})$ HAS A GLOBAL MINIMUM, SAY
AT $\underline{y}_0 \in V$. IF $\underline{y} \in V$, $(\underline{x}-\underline{y})\cdot(\underline{x}-\underline{y}) \geq (\underline{x}-\underline{y}_0)\cdot(\underline{x}-\underline{y}_0)$.

WE CLAIM $\underline{x} - \underline{y}_0$ IS PERPENDICULAR TO ALL $\underline{v} \in V$ SO $\underline{y}_0 = \underline{x}^{\parallel}$, $\underline{x} - \underline{y}_0 = \underline{x}^{\perp}$.

SUPPOSE OTHERWISE AND CONSIDER THE FUNCTION

$$\|\underline{x} - \underline{y}_0 - t\underline{v}\|^2, \quad t \in \mathbb{R}.$$

$$= (\underline{x}-\underline{y}_0 - t\underline{v})\cdot(\underline{x}-\underline{y}_0-t\underline{v}) = \|\underline{x}-\underline{y}_0\|^2 - 2t\,(\underline{x}-\underline{y}_0)\cdot\underline{v} + t^2\|\underline{v}\|^2.$$

TO BE MINIMIZED AT $t=0$: $(\underline{x}-\underline{y}_0)\cdot\underline{v} = 0$. $\Rightarrow (\underline{x}-\underline{y}_0)\perp V$.

SUPPOSE $\underline{x} = \underline{x}_1^{\parallel} + \underline{x}_1^{\perp}$

$= \underline{x}_2^{\parallel} + \underline{x}_2^{\perp}$.

THEN $\underline{x}_1^\parallel - \underline{x}_2^\parallel = \underline{x}_2^\perp - \underline{x}_1^\perp$. $= 0$.

V $\quad \perp$ TO ALL $\underline{v}\in V$.

$\|\underline{x}_1^\parallel - \underline{x}_2^\parallel\|^2 = (\underline{x}_1^\parallel - \underline{x}_2^\parallel)\cdot(\underline{x}_2^\perp - \underline{x}_1^\perp) = 0$. $\quad \underline{x}_1^\perp = \underline{x}_2^\perp$, $\underline{x}_1^\parallel = \underline{x}_2^\parallel$.

THEOREM: IF $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ IS AN ORTHONORMAL BASIS FOR V,

$$\mathrm{PROJ}_V\, \underline{x} = \underline{x}^\parallel = (\underline{u}_1\cdot\underline{x})\underline{u}_1 + (\underline{u}_2\cdot\underline{x})\underline{u}_2 + \dots + (\underline{u}_n\cdot\underline{x})\underline{u}_n.$$

PROOF: CONSIDER

$$(\underline{x} - \underline{x}^{\parallel}) \cdot \underline{v}, \quad \underline{v} \in V.$$

$$= \left(\underline{x} - \left((\underline{u}_1 \cdot \underline{x})\underline{u}_1 + (\underline{u}_2 \cdot \underline{x})\underline{u}_2 + \dots + (\underline{u}_n \cdot \underline{x})\underline{u}_n\right)\right) \cdot \underline{v}.$$

SUPPOSE FIRST $\underline{v} = \underline{u}_i$. THIS IS $\underline{x} \cdot \underline{u}_i - (\underline{u}_i \cdot \underline{x}) = 0$

IF $\underline{v} = c_1\underline{u}_1 + \dots + c_n\underline{u}_n$ FOLLOW BY LINEARITY. THUS $\underline{x} - \underline{x}^{\parallel} \perp V$. $\square$

THEOREM. IF $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ ORTHONORMAL BASIS OF $\mathbb{R}^n$,

$$\underline{x} = (\underline{x} \cdot \underline{u}_1)\underline{u}_1 + (\underline{x} \cdot \underline{u}_2)\underline{u}_2 + \dots + (\underline{x} \cdot \underline{u}_n)\underline{u}_n$$

AND $\|\underline{x}\|^2 = (\underline{x} \cdot \underline{u}_1)^2 + (\underline{x} \cdot \underline{u}_2)^2 + \dots + (\underline{x} \cdot \underline{u}_n)^2$ (PYTHAGOREAN THEOREM IN $\mathbb{R}^n$).

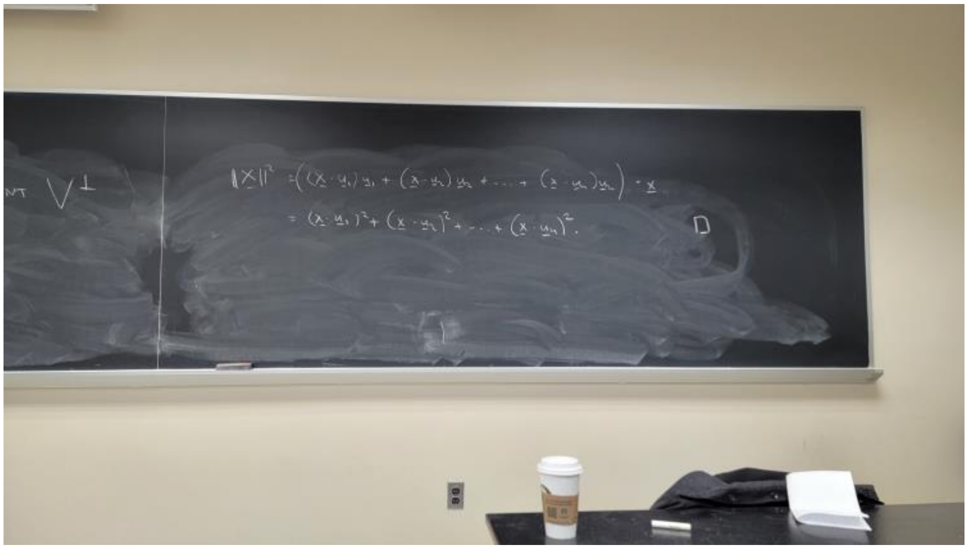
PROOF: LET $\tilde{\underline{x}} = (\underline{x} \cdot \underline{u}_1)\underline{u}_1 + \dots + (\underline{x} \cdot \underline{u}_n)\underline{u}_n$.

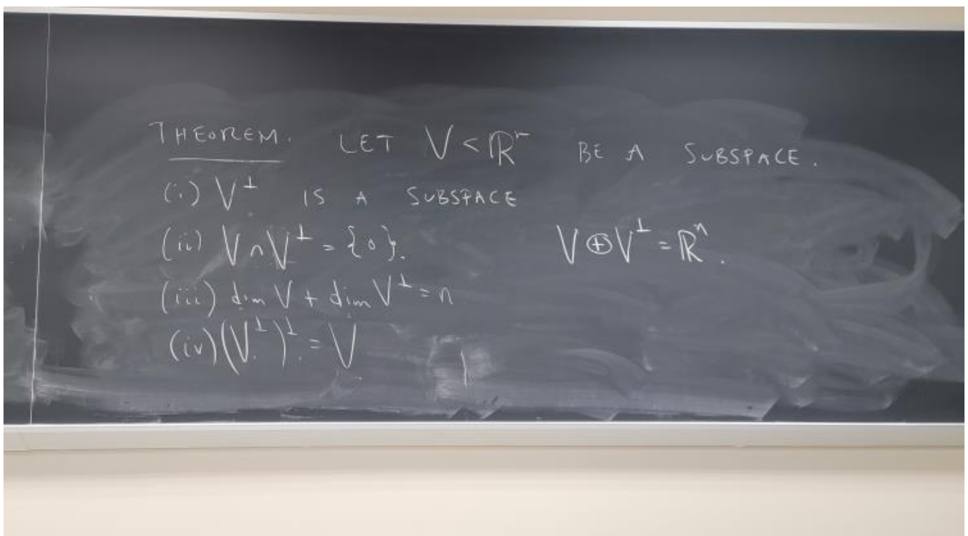
THEN $\tilde{\underline{x}} \cdot \underline{u}_i = \underline{x} \cdot \underline{u}_i$.

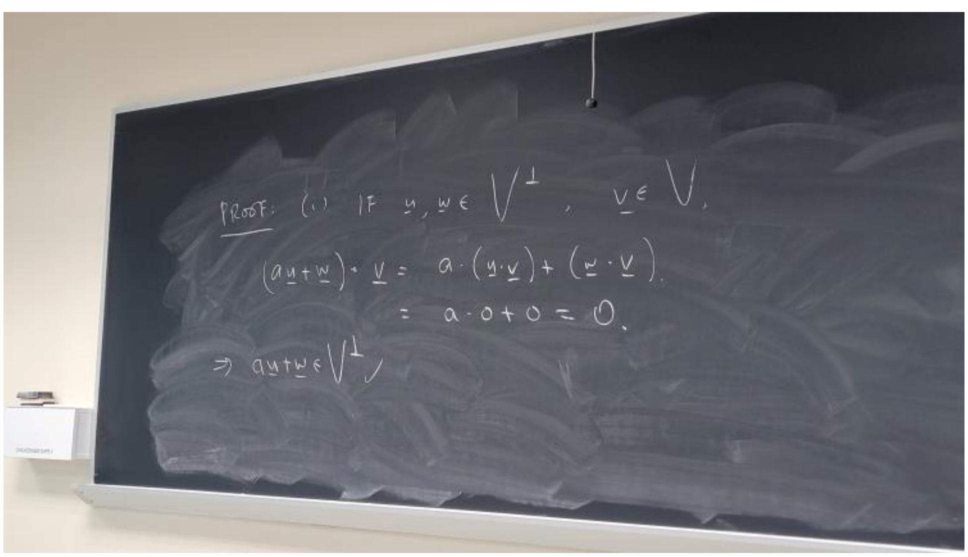
SO $(\underline{x} - \tilde{\underline{x}}) \cdot \underline{v} = 0$ ALL $\underline{v} \in \mathbb{R}^n$ $(\underline{v} = c_1\underline{u}_1 + \dots + c_n\underline{u}_n)$.

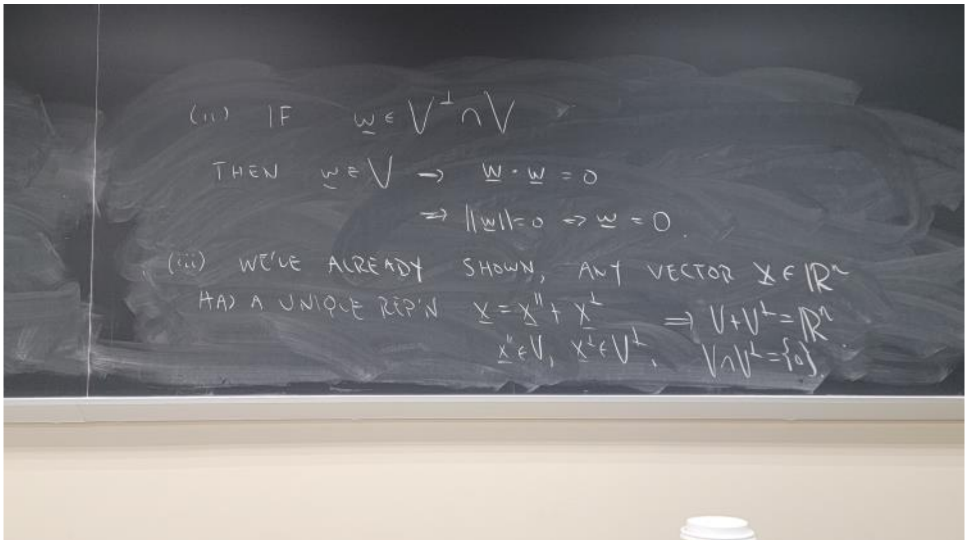
$\Rightarrow (\underline{x} - \tilde{\underline{x}}) \cdot (\underline{x} - \tilde{\underline{x}}) = 0 \Rightarrow \|\underline{x} - \tilde{\underline{x}}\|^2 = 0$

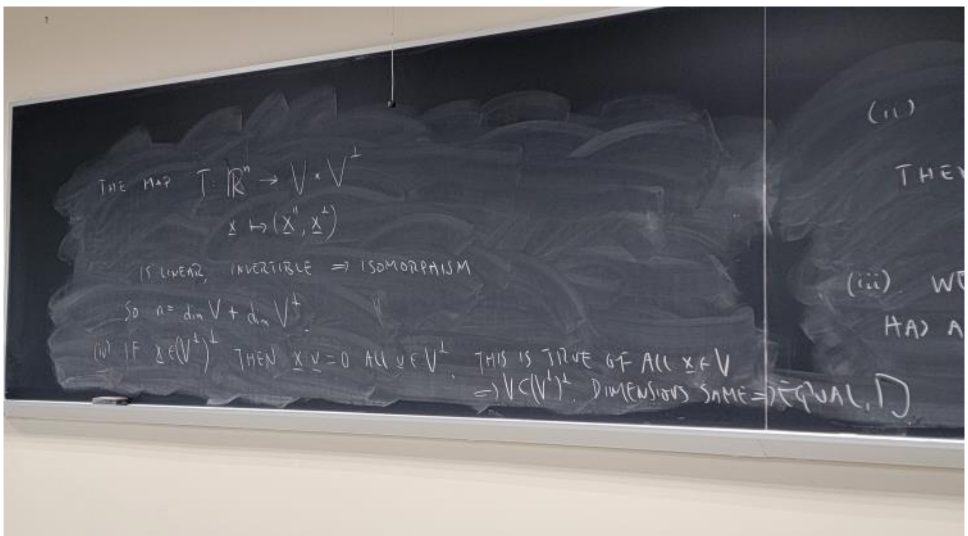
$\Rightarrow \underline{x} = \tilde{\underline{x}}$.


NT $V^\perp$
$\|\underline{x}\|^2 = \left((\underline{x}\cdot \underline{y}_1)\underline{y}_1 + (\underline{x}\cdot \underline{y}_2)\underline{y}_2 + \dots + (\underline{x}\cdot \underline{y}_n)\underline{y}_n\right)\cdot \underline{x}$
$= (\underline{x}\cdot\underline{y}_1)^2 + (\underline{x}\cdot\underline{y}_2)^2 + \dots + (\underline{x}\cdot\underline{y}_n)^2.$
$\square$


THEOREM. LET $V < \mathbb{R}^n$ BE A SUBSPACE.
(i) $V^\perp$ IS A SUBSPACE
(ii) $V \cap V^\perp = \{0\}$.
$V \oplus V^\perp = \mathbb{R}^n$.
(iii) $\dim V + \dim V^\perp = n$
(iv) $(V^\perp)^\perp = V$


PROOF: (i) IF $\underline{u}, \underline{w} \in V^\perp$, $\underline{v} \in V$,
$(a\underline{u} + \underline{w}) \cdot \underline{v} = a\cdot(\underline{u}\cdot\underline{v}) + (\underline{w}\cdot\underline{v})$
$= a \cdot 0 + 0 = 0.$
$\Rightarrow a\underline{u} + \underline{w} \in V^\perp$ ✓


(ii) IF $\underline{w} \in V^\perp \cap V$
THEN $\underline{w} \in V \rightarrow \underline{w} \cdot \underline{w} = 0$
$\Rightarrow \|\underline{w}\| = 0 \Leftrightarrow \underline{w} = 0$.
(iii) WE'VE ALREADY SHOWN, ANY VECTOR $\underline{x} \in \mathbb{R}^n$
HAD A UNIQUE REP'N $\underline{x} = \underline{x}^{\parallel} + \underline{x}^{\perp}$ $\Rightarrow V + V^\perp = \mathbb{R}^n$
$\underline{x}^{\parallel} \in V$, $\underline{x}^\perp \in V^\perp$. $V \cap V^\perp = \{0\}$.


THE MAP $T: \mathbb{R}^n \rightarrow V \times V^\perp$
$\underline{x} \mapsto (\underline{x}^{\parallel}, \underline{x}^{\perp})$
IS LINEAR, INVERTIBLE $\Rightarrow$ ISOMORPHISM
SO $n = \dim V + \dim V^\perp$.
(iv) IF $\underline{x} \in (V^\perp)^\perp$ THEN $\underline{x} \cdot \underline{v} = 0$ ALL $\underline{v} \in V^\perp$. THIS IS TRUE OF ALL $\underline{x} \in V$
$\Rightarrow V \subset (V^\perp)^\perp$. DIMENSIONS SAME $\Rightarrow$ EQUAL. $\square$
(ii)
THEN
(iii) WE
HAD A